Fluid-particle separation in a random flow described by the telegraph model

Gregory Falkovich and Marco Martins Afonso*

Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot, Israel
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We study the statistics of the relative separation between two fluid particles in a spatially smooth and temporally random flow. The Lagrangian strain is modeled by a telegraph noise, which is a stationary random Markov process that can only take two values with known transition probabilities. The simplicity of the model enables us to write closed equations for the interparticle distance in the presence of a finite-correlated noise. In one dimension, we are able to find analytically the long-time growth rates of the distance moments and the senior Lyapunov exponent, which consistently turns out to be negative. We also find the exact expression for the Cramér function and show that it satisfies the fluctuation relation (for the probability of positive and negative entropy production) despite the time irreversibility of the strain statistics. For the two-dimensional incompressible isotropic case, we obtain the Lyapunov exponent (positive) and the asymptotic growth rates of the moments in two opposite limits of fast and slow strain. The quasideterministic limit (of slow strain) turns out to be singular, while a perfect agreement is found with the already-known δ-correlated case.

I. INTRODUCTION

Understanding the statistics of the relative separation between two fluid particles in a generic random flow is a difficult task of significant importance [1–5]. Here we treat small distances between particles, where the flow can be considered spatially smooth (so-called Batchelor regime [6]). We thus consider a general problem of dynamical systems where the linearized equation for the (infinitesimal) relative separation between two fluid particles, $\mathbf{R}(t)$, takes the form

$$d_t\mathbf{R}(t) = \sigma(t)\mathbf{R}(t). \tag{1}$$

Here the matrix of the velocity derivatives $\sigma(t)$ is called strain, it is taken in the comoving (Lagrangian) frame and its statistics is supposed to be given. At times much exceeding the correlation time of the strain, general properties of systems with a dynamical chaos (called Lagrangian chaos in fluid mechanics) are described by the multifractal spectrum $\gamma(n)$ of the growth rates of the moments:

$$\langle R^n(t)\rangle \sim e^{\gamma(n)t}. \tag{2}$$

The function $\gamma(n)$ is convex, its derivative at zero is called the senior Lyapunov exponent,

$$\lambda \equiv \lim_{t\to\infty}[\langle \ln R(t)\rangle/t] = \gamma'(0). \tag{3}$$

Apart from that, not much is known about the general properties of the function $\gamma(n)$. This is why few particular solvable cases are of much importance [4,7]. For the statistically isotropic strain, only three cases were found amenable to an analytic treatment [4]: the completely solvable short-correlated case in arbitrary dimension [where $\gamma(n)$ is a parabola], and two partially solvable cases, the long-correlated case in two dimensions (where the Lyapunov exponent and its dispersion are found [8]) and the large-dimensionality case (where the second and the fourth moments are found

together with the different-time correlation functions [9]).

In this paper we describe another situation which allows exact analytical calculations: the telegraph-noise model for the strain [10]. This model enables one to describe analytically the dependence of $\gamma(n)$ on the correlation time of the strain. We shall study both the one-dimensional (1D) (compressible) and the incompressible 2D cases. The telegraph noise is a stationary random process, $\alpha(t)$, which satisfies the Markov property and only takes two values, a_1 and a_2. The probability, per unit time, of passing from the latter state to the former (or vice versa) is given by ν_1 (ν_2, respectively). In what follows, we shall consider some special cases, for which simple formulas hold.

(i) If $a_1 = -a_2$, then the process itself is stochastic but its square is deterministic, keeping the constant value $\alpha^2(t) = a^2$ (with $a = |a_1| = |a_2|$). We shall denote such processes with a star, e.g., $\alpha^*(t)$.

(ii) If the average $\langle\alpha\rangle = (\nu_1 a_1 + \nu_2 a_2)/\nu$ vanishes (with $\nu = \nu_1 + \nu_2$), then the autocorrelation takes the form $\langle\alpha(t)\alpha(t')\rangle = e^{-\nu|t-t'|}(\nu_1 a_1^2 + \nu_2 a_2^2)/\nu$. For any analytical functional $F[\alpha]$ (function of time), a simple "formula of differentiation" [10] then holds in this case:

$$d_t\langle\alpha(t)F[\alpha]\rangle = \langle\alpha(t)d_tF[\alpha]\rangle - \nu\langle\alpha(t)F[\alpha]\rangle. \tag{4}$$

We shall denote such processes with a tilde, e.g., $\tilde{\alpha}(t)$. Note that it is always possible to reduce to this case from a generic telegraph noise [in particular from one $\alpha^*(t)$], simply by subtracting from the latter its mean value.

(iii) If both the properties in (i) and (ii) hold simultaneously, the maximum level of simplification is reached. For the sake of simplicity, we shall denote such processes not with $\tilde{\alpha}^*(t)$ but rather with $\xi(t)$.

This type of noise has often been used to mimic more general colored noise in stochastic theories and has also been applied, e.g., to optics (see [11] and references therein). An interesting application of the "telegraph concept" to turbulence can be found in [12]. In [10] a possible extension to the case of a multivalued telegraph process is discussed. While

*marcomar@fisica.unige.it

the telegraph model is still far from any realistic flow, it is one step closer to reality than the previous (short-correlated) models that allowed for an analytical treatment of the two-point separation [4]. Our results provide, in our opinion, a first step in the analytic description of the role played by the strain correlation time in the statistics of the fluid-particle pair separation.

The paper is organized as follows: in Sec. II we will study the 1D case. In Sec. III we will move to 2D, describing the general incompressible isotropic case; the latter will be analyzed at first in its complete form, then (respectively, in Secs. III A and III B) by means of a simplified dynamics and eventually in the short-correlated limit. The quasideterministic limit is also shown in Sec. III C. Conclusions follow in Sec. IV.

II. 1D CASE

The simplest meaningful model, making use of the telegraph noise, for the 1D case, consists in assuming a strain function with the property (i), i.e.,

$$\sigma(t) = \alpha^*(t). \tag{5}$$

Indeed, the problem is compressible locally, i.e., within every single realization of the noise, but for fluid-mechanical applications we want the average (1D) "volume" $\langle R(t) \rangle$ to be conserved. This means that we have to impose the constraint

$$\gamma(1) = 0. \tag{6}$$

From Eqs. (1) and (5), it is straightforward to obtain the system of the two first-order ordinary differential equations which describe the time evolution of moments. It is indeed sufficient to multiply the starting equation by the noise itself (taken at the same time) in order to obtain a closed system, thanks to the simplicity of the telegraph form. Namely, if we define $\alpha_0 \equiv \langle \alpha^*(t) \rangle$ and rewrite accordingly $\alpha^*(t) = \widetilde{\alpha}(t) + \alpha_0$, we obtain

$$d_t R^n(t) = n\alpha^*(t)R^n(t) = n[\widetilde{\alpha}(t) + \alpha_0]R^n(t) \tag{7}$$

and thus

$$d_t \langle R^n(t) \rangle = n\langle \widetilde{\alpha}(t)R^n(t) \rangle + n\alpha_0 \langle R^n(t) \rangle. \tag{8}$$

We must then look for the evolution equation of $\langle \widetilde{\alpha}(t)R^n(t) \rangle$, which is readily done by exploiting Eq. (4),

$$d_t \langle \widetilde{\alpha}(t)R^n(t) \rangle = \langle \widetilde{\alpha}(t)d_t R^n(t) \rangle - \nu \langle \widetilde{\alpha}(t)R^n(t) \rangle. \tag{9}$$

Taking Eq. (7) into account, we obtain $\langle \widetilde{\alpha}(t)d_t R^n(t) \rangle = na^2 \langle R^n(t) \rangle - n\alpha_0^2 \langle R^n(t) \rangle - n\alpha_0 \langle \widetilde{\alpha}(t)R^n(t) \rangle$, where we made use of the property stated in (i). Consequently, Eq. (9) is rewritten as

$$d_t \langle \widetilde{\alpha}(t)R^n(t) \rangle = n(a^2 - \alpha_0^2)\langle R^n(t) \rangle - (\nu + n\alpha_0)\langle \widetilde{\alpha}(t)R^n(t) \rangle. \tag{10}$$

Equations (8) and (10) constitute the system we were looking for. It is easy to recast the latter as a second-order differential equation for the quantity $\langle R^n(t) \rangle$ alone, which can then be solved exactly:

$$d_t^2 \langle R^n(t) \rangle + \nu d_t \langle R^n(t) \rangle - n(na^2 + \nu\alpha_0)\langle R^n(t) \rangle = 0,$$

$$\langle R^n(t) \rangle = b_n \exp\{-(\sqrt{\nu^2 + 4n^2a^2 + 4n\nu\alpha_0} + \nu)t/2\} + \beta_n \exp\{(\sqrt{\nu^2 + 4n^2a^2 + 4n\nu\alpha_0} - \nu)t/2\},$$

$$\gamma(n) = (\sqrt{\nu^2 + 4n^2a^2 + 4n\nu\alpha_0} - \nu)/2. \tag{11}$$

Constraint (6) applied to Eq. (11) implies

$$\alpha_0 = -\frac{a^2}{\nu} \Rightarrow \gamma(n) = \frac{\sqrt{\nu^2 + 4a^2n(n-1)} - \nu}{2}. \tag{12}$$

This is an even function of $n - 1/2$, it is convex and the asymptotics are $\gamma(n) \sim \pm an$ at $n \to \pm\infty$.

The mean value α_0 is negative, but its modulus cannot exceed a, which means that only situations with a/ν (the analog of the Kubo number) smaller than unity are physically relevant. Another interesting consequence is the relation $\nu_1 - \nu_2 = -a$, which reduces from three to two the number of free parameters in the original definition (5) of the strain. This means that the lower level of the noise persists longer: in the Lagrangian frame, indeed, particles spend longer time in contracting regions than in expanding ones, thus enhancing the weight of the former in the statistics. Accordingly, the Lyapunov exponent, computed as the derivative of Eq. (12) taken in the origin, turns out to be negative as is required by the general theory [13,14]:

$$\lambda = \gamma'(0) = -a^2/\nu. \tag{13}$$

The joint probability density function $P(R,t)$ can also be investigated. It satisfies the second-order partial differential equation [10]

$$\partial_t^2 P + \nu\partial_t P - a^2 \partial_R[R\partial_R(RP)] + a^2 \partial_R(RP) = 0. \tag{14}$$

Equation (14) has no stationary solutions, as both $P = \text{const}$ and $P \propto R^{-1}$ are non-normalizable. The asymptotic solution has a large-deviation form

$$P(R,t) \overset{t \to +\infty}{\sim} e^{-tH(X)}, \tag{15}$$

where $X \equiv t^{-1} \ln[R/R(0)]$ can only belong [because of Eq. (1)] to the interval $[-a,a]$. The Cramér function $H(X)$ is simply the Legendre transform of $\gamma(n)$,

$$H(X) = \frac{\nu}{2} + \frac{X}{2} - \frac{\sqrt{(\nu^2 - a^2)(a^2 - X^2)}}{2a}. \tag{16}$$

It is a convex function, which exactly satisfies the Evans-Searles-Gallavotti-Cohen fluctuation relation $H(X) - H(-X) = X$ [15,16] despite the fact that the strain statistics is not time reversible [17]. The function vanishes quadratically at the minimum, represented by the Lyapunov exponent, $H(X) \overset{X \sim \lambda}{\sim} (X - \lambda)^2 \nu^3/4a^2(\nu^2 - a^2)$, and it approaches vertically the boundaries of its compact domain:

$$H(X) \overset{X \sim \pm a}{\sim} \frac{\nu \pm a}{2} - \sqrt{\frac{\nu^2 - a^2}{2a}}\sqrt{a \mp X}.$$

Plugging Eq. (16) into Eq. (15), we obtain the final result

$$P(R,t) \overset{t\to+\infty}{\sim} |R|^{-1/2} \exp\{(\sqrt{(1-a^2/\nu^2)\{1-\ln^2[R/R(0)]/a^2t^2\}} - 1)\nu t/2\}.$$

It is worth mentioning what happens if one does not impose the constraint (6) and assumed a strain $\sigma(t) = \xi(t)$ with the property described in (iii). In this case, $\gamma(n) = (\sqrt{\nu^2 + 4n^2a^2} - \nu)/2$ and the Lyapunov exponent would vanish. The Cramér function would be $H(X) = \nu/2 - \nu\sqrt{a^2 - X^2}/2a$, so that

$$P(R,t) \overset{t\to\infty}{\sim} |R|^{-1} \exp[-(1 - \sqrt{1 - \ln^2[R/R(0)]/a^2t^2})\nu t/2]$$

would be the solution to the equation for the joint probability distribution, similar to Eq. (14) but without the last term.

III. 2D CASE

In the 2D case, it is possible to consistently impose the incompressibility constraint locally, for every single realization of the noise. Since this case is most interesting for applications, we shall only consider this situation, which implies the strain matrix to be traceless.

Consider a general statistically isotropic flow described by the strain matrix

$$\sigma_{ij}(t) = \begin{pmatrix} \xi_I(t) & \xi_{II}(t) + \sqrt{2}\xi_{III}(t) \\ \xi_{II}(t) - \sqrt{2}\xi_{III}(t) & -\xi_I(t) \end{pmatrix}.$$

The noises $\xi_I(t)$, $\xi_{II}(t)$, and $\xi_{III}(t)$ are independent of one another but share the properties in (iii) with the same coefficients a and ν, which can now range from 0 to ∞ independently. Differently from the 1D case, we will see that this is enough to mimic a realistic situation. The matrix σ thus turns out to be singular and nilpotent: $\det \sigma = 0 = \sigma^2$.

By means of simple manipulations, analog to those described for the 1D case, one can write down a closed system for the moments of the components $r_{n,k}(t) \equiv R_1^k(t)R_2^{n-k}(t)$ at every n, in the form

$$d_t\langle\mathcal{R}_\iota^{(n)}\rangle = A_{\iota\kappa}^{(n)}\langle\mathcal{R}_\kappa^{(n)}\rangle. \tag{17}$$

The column vector $\mathcal{R}^{(n)}$ has dimension $8(n+1)$, being made up of the following: the $n+1$ components of $r_{n,\bullet}$ itself, the $3(n+1)$ components of $r_{n,\bullet}$ times one noise, the $3(n+1)$ components of $r_{n,\bullet}$ times two (distinct) noises, and finally the $n+1$ components of $r_{n,\bullet}$ times all the three noises. No other quantity is required to close the system, because as soon as a noise appears as square it becomes a deterministic quantity and can be taken out of the statistical average. The matrix $A^{(n)}$ is thus of order $8(n+1) \times 8(n+1)$ and turns out to be dependent only on a and ν. The number of noises sharply increases in higher dimensions, thus strongly enhancing the number of components in $\mathcal{R}^{(n)}$: this is why we confine ourselves to the 2D case.

To study the long-time evolution of $\langle r_{n,k}\rangle$, one should identify the eigenvalue of $A^{(n)}$ with the largest positive real part. This is not an easy task from the numerical point of view for growing n, and it does not look feasible analytically for a generic n, even if the matrix itself can be easily written

down as a function of n (the details can be found in [18] where, in particular, the whole spectrum of eigenvalues is found for $n=1$ and $n=2$).

The evolution of the linear components $\langle R_1\rangle$ and $\langle R_2\rangle$ is straightforward to describe: the largest eigenvalue is 0, which means that they do not show any exponential growth at large times; however, as they can change sign, this does not yield any information on $\langle R\rangle$, i.e., on $\gamma(1)$. The situation is more interesting for the quadratic components (R_1^2, R_1R_2, R_2^2), because in this case the largest eigenvalue is given by

$$\gamma(2) = -\nu + \frac{\nu^2}{\sqrt[3]{216a^2\nu + 3\sqrt{5184a^4\nu^2 - 3\nu^6}}} + \frac{\sqrt[3]{216a^2\nu + 3\sqrt{5184a^4\nu^2 - 3\nu^6}}}{3}, \tag{18}$$

which is always positive (i.e., describes an exponential growth of $\langle R^2\rangle = \langle R_1^2\rangle + \langle R_2^2\rangle$) and behaves asymptotically as

$$\gamma(2) = \begin{cases} 8\dfrac{a^2}{\nu} - 96\dfrac{a^4}{\nu^3}\left[1 + O\left(\dfrac{a}{\nu}\right)^2\right], & \text{for } \dfrac{a}{\nu} \to 0, \\ 2\sqrt[3]{2a^2\nu} - \nu\left[1 + O\left(\dfrac{\nu}{a}\right)^{2/3}\right], & \text{for } \dfrac{a}{\nu} \to \infty. \end{cases} \tag{19}$$

A. Exact reduced dynamics

Since the general process is computationally very demanding, we succeeded to find a reduced set of variables invariant with respect to the evolution, i.e., a subset of components of $\mathcal{R}^{(n)}$ such that the corresponding rows in $A^{(n)}$ are nonzero only in columns whose index has been taken into account in the reduced subset. Consider even n,

$$R^n = (R_1^2 + R_2^2)^{n/2} = \sum_{k=0,2,\ldots,n} \binom{n/2}{k/2} r_{n,k}, \tag{20}$$

by definition for positive even n, thus the values $\gamma(n)$ can be extracted from the knowledge of $n/2+1$ of the first $n+1$ components in $\mathcal{R}^{(n)}$. On the contrary, this is not the case for the odd n's, which behave similarly to non-natural n's because they involve nonlinear operations (in this case a square root) that do not commute with the statistical averaging.

In particular, we analyzed the values $n=4$, $n=6$, and $n=8$. Adopting a technique described in [18], one obtains matrices of order 5, 7, and 9, respectively (i.e., $n+1$), whose eigenvalues with largest positive real part behave asymptotically as

$$\gamma(4) \sim \begin{cases} 24\dfrac{a^2}{\nu} - 576\dfrac{a^4}{\nu^3} & \text{for } \dfrac{a}{\nu} \to 0, \\ 2\sqrt[5]{36a^4\nu} & \text{for } \dfrac{a}{\nu} \to \infty, \end{cases}$$

$$\gamma(6) \sim \begin{cases} 48\dfrac{a^2}{\nu} - 2016\dfrac{a^4}{\nu^3} & \text{for } \dfrac{a}{\nu} \to 0, \\[2ex] 2\sqrt[7]{1800a^6\nu} & \text{for } \dfrac{a}{\nu} \to \infty, \end{cases}$$

$$\gamma(8) \sim \begin{cases} 80\dfrac{a^2}{\nu} - 5280\dfrac{a^4}{\nu^3} & \text{for } \dfrac{a}{\nu} \to 0, \\[2ex] 2\sqrt[9]{181440a^8\nu} & \text{for } \dfrac{a}{\nu} \to \infty. \end{cases}$$

These relations provide us with three points of the curve $\gamma(n)$ in the asymptotic conditions, to which three more can be added: besides the already-mentioned $\gamma(2)$, one should indeed remember that both $\gamma(0)$ and $\gamma(-2)$ (in 2D) must vanish [2,4,19]. A simple extrapolation then suggests the following asymptotic behavior:

$$\gamma(n) \sim \begin{cases} G(n)\dfrac{a^2}{\nu} + \mathcal{G}(n)\dfrac{a^4}{\nu^3} & \text{for } \dfrac{a}{\nu} \to 0, \\[2ex] g(n)a^{n/(n+1)}\nu^{1/(n+1)} & \text{for } \dfrac{a}{\nu} \to \infty. \end{cases} \tag{21}$$

Note that the noise correlation time is $T = 1/\nu$. The nth growth rate grows linearly with T when the latter is small and decays as $T^{-1/(n+1)}$ when it is large.

The coefficient $G(n)$ can be proven rigorously to be $n(n+2)$ (see Sec. III B), while such a proof does not look feasible for $\mathcal{G}(n)$. However, starting from the quadratic form of the former, one can guess a fourth-order polynomial for the latter, $\mathcal{G}(n) = 3n(n+2)(n^2+2n+8)/4$, and the correctness of this expression seems to be confirmed by the fact that it satisfies the six conditions in $n = -2, 0, 2, 4, 6, 8$ despite possessing only five degrees of freedom. As the following coefficients in the power expansion at small a/ν are expected to show higher powers of n, such an expansion is not uniform in the sense that it must fail for some (large enough) value of n. On the contrary, the coefficient $g(n)$ must originate from some combinatorics, but has not been identified yet, while the presence of noninteger powers of a and ν suggests that the limit $a/\nu \to \infty$ is singular. From Eq. (21), exploiting the knowledge of $G(n)$ and the vanishing of $g(0)$, we obtain the asymptotic behavior of the Lyapunov exponent,

$$\lambda = \begin{cases} 2\dfrac{a^2}{\nu} & \text{for } \dfrac{a}{\nu} \to 0, \\[2ex] g'(0)\nu & \text{for } \dfrac{a}{\nu} \to \infty. \end{cases} \tag{22}$$

B. Short-correlated case

The results obtained in the previous section are exact, but they have been obtained for particular values of n, and the generalization of the coefficients (21) as functions of n was empirical. There is however at least one limit in which rigorous analytical results can be obtained: it is provided by the short-correlated case, corresponding to small a/ν. We re-

mind the reader that in the exactly δ-correlated case [4,5] the probability distribution of $R(t)$ is logarithmic normal and the Cramér function is quadratic. Note that the correct limit is such that $a/\nu \to 0$, but a^2/ν is finite. The key point in this limit is that the leading behavior can be extracted rigorously, for any natural n, from a restricted dynamics, namely taking into account only the first $4(n+1)$ components in $\mathcal{R}^{(n)}$ (corresponding to the average of the coordinates $r_{n,k}$, and of the latter times one noise). The details of the calculations can be found in [18]; here we only mention that, upon introducing the small-Kubo Taylor expansion, we face a *degenerate perturbation theory*. Indeed, the eigenvalues corresponding to a fixed n are not definitely separate for $a/\nu \to 0$, but rather some of them vanish in this limit, and we have to identify the one which keeps the largest positive real part in this process. The result of this analysis is that in the evolution equation for $\langle r_{n,k} \rangle$ one should account for the complete form, while for $\langle \xi \cdot r_{n,k} \rangle$ one can neglect all source terms involving two different noises: the system thus closes at this stage, reducing the order of the corresponding matrix from the initial $8(n+1)$ to $4(n+1)$. In other words, only the upper-left quarter of the matrix $A^{(n)}$ is relevant. The problem can now easily be recast as a system of $n+1$ second-order differential equations for $r_{n,k}$. Indeed, we have

$$d_t^2 r_{n,k} = kR_1^{k-1}R_2^{n-k}d_t^2R_1 + (n-k)R_1^kR_2^{n-k-1}d_t^2R_2$$
$$+ k(k-1)R_1^{k-2}R_2^{n-k}(d_tR_1)^2 + (n-k)(n-k-1)$$
$$\times R_1^kR_2^{n-k-2}(d_tR_2)^2 + 2k(n-k)R_1^{k-1}R_2^{n-k-1}d_tR_1d_tR_2,$$

and, keeping into account the aforementioned simplification in the equations for $\langle \xi \cdot r_{n,k} \rangle$,

$$d_t^2\langle r_{n,k}\rangle + \nu d_t\langle r_{n,k}\rangle = a^2[(n^2 + 6k^2 - 6nk - n)\langle r_{n,k}\rangle + 3k(k-1)$$
$$\times \langle r_{n,k-2}\rangle + 3(n-k)(n-k-1)\langle r_{n,k+2}\rangle]$$
$$\equiv a^2 M_{kk'}^{(n)}\langle r_{n,k'}\rangle. \tag{23}$$

The matrix $M^{(n)}$, defined by the right-hand side of Eq. (23), is *a priori* of order $(n+1) \times (n+1)$. However, it is apparent from Eq. (23) that, for a fixed n, the dynamics of the components of $\langle r_{n,k}\rangle$ with even k are independent of those with odd k. Therefore, as our main goal is to reconstruct objects such as Eq. (20), we can just focus on subset $\hat{M}^{(n)}$ of $M^{(n)}$ consisting of the lines with even k, and thus reduce to a $(n/2+1) \times (n/2+1)$ problem, simply neglecting the lines corresponding to odd k. In other words, we rewrite Eq. (23) as

$$d_t^2\langle r_{n,2k}\rangle + \nu d_t\langle r_{n,2k}\rangle = a^2 \sum_{k'=0,2,\ldots,n} M_{2k,k'}^{(n)}\langle r_{n,k'}\rangle = a^2 \hat{M}_{kk'}^{(n)}$$
$$\times \langle r_{n,2k'}\rangle. \tag{24}$$

Moreover, it is easy to show that the largest-positive-real-part eigenvalue μ_n (divided by a^2/ν) that we are looking for must belong to the spectrum of the subset matrix $\hat{M}^{(n)}$. Such subset matrix is tridiagonal central symmetric, but not symmetric, which means that left and right eigenvectors differ. It is a simple task to prove that the value $n(n+2)$ [guessed for

$\mu_2/(a^2/\nu)$ asymptotically from the previous section] is actually an eigenvalue, but it is not possible to use a variational method to show that it has the largest positive real part. However, this is not needed, because the row vector built with the $n/2+1$ components of the binomial coefficient $\binom{n/2}{k}$ (for $k=0,1,\ldots,n/2$) happens to be the left eigenvector of $\hat{M}^{(n)}$ corresponding to the eigenvalue $n(n+2)$. Therefore, using Eqs. (20) and (24), one has (for details see [18])

$$d_t^2\langle R^n\rangle + \nu d_t\langle R^n\rangle = n(n+2)a^2\langle R^n\rangle.$$

This implies

$$\mu_n \overset{a/\nu\to0}{\sim} \frac{\sqrt{\nu^2 + 4n(n+2)a^2} - \nu}{2} = n(n+2)\frac{a^2}{\nu}\left[1 + O\left(\frac{a}{\nu}\right)^2\right],$$

as expected. Notice that the next-leading correction cannot be captured at this stage, because of the approximation introduced previously. To take it into account correctly, one should reformulate the analysis performed in this section, including also quantities such as $\langle\xi_\bullet\xi_\circ r_{n,k}\rangle$, but however excluding $\langle\xi_I\xi_{II}\xi_{III}r_{n,k}\rangle$, i.e., considering a reduced dynamics in the first $7(n+1)$ components of $\mathcal{R}^{(n)}$.

It is worth mentioning that an analysis such as the previous one, but applied to odd n's, leads to

$$\mu_n \overset{a/\nu\to0}{\sim} (n-1)(n+3)\frac{a^2}{\nu},$$

in accordance, e.g., with the exact result $\mu_1=0$ found with the complete dynamics. However, as already pointed out, such values are not related to the curve $\gamma(n)$.

C. Deterministic limit in 2D

The deterministic case corresponds to $\nu=0$, when the noise $\xi(t)$ takes a constant value $\pm a$. In 2D, one obtains $d_t^2R_1=0=d_t^2R_2$ (for any of the $2^3=8$ possible combinations of the noise signs), thus both components are linear in time and $R^n \overset{t\to\infty}{\sim} t^n$. Consequently, a power-law temporal dependence is

found for the separation and $\lambda=0$. It is worth noticing that this result is due to the fact that, with our choice, $\sigma^2=0$, and is characteristic of the 2D isotropic situation. Therefore the limit is singular: an exponential growth is expected in general for finite a/ν, but a power law is found when this ratio is infinite. As a result, perturbation theory and expansion in integer powers of ν/a does not work. In Eq. (21) this is reflected by the presence of noninteger powers and the difference between the limits $a\to\infty$ and $\nu\to0$.

IV. CONCLUSIONS

By assuming the telegraph-noise model for the velocity gradient (or strain matrix), we were able to carry out analytical computations and to obtain several results on the separation between two fluid particles. Focusing on smooth flows (Batchelor regime), we first analyzed the one-dimensional compressible case, finding explicit expressions for the long-time evolution of the interparticle-distance moments (12), the Lyapunov exponent (13), and the Cramér function (16). Then we concentrated on the two-dimensional incompressible isotropic case, where a thorough analysis on the complete dynamics was made for the evolution of linear ($n=1$) and quadratic ($n=2$) components. Due to high computational cost, at higher n we focused on a restricted, though exact, dynamics: however, only specific values of n could be studied in this way, leading to the extrapolations (21) and (22). Such guess was rigorously proved in the quasi-δ-correlated limit, for which approximated equations were introduced.

We believe that the present paper represents an interesting example of the use of a colored noise for which the closure problem can be solved analytically. In particular, we established the dependence of the distance growth rates and of the Lyapunov exponents on the correlation time.

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